Note on the energy of regular graphs *

Xueliang Li, Yiyang Li and Yongtang Shi
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
lxl@nankai.edu.cn; liycldk@mail.nankai.edu.cn; shi@nankai.edu.cn

Abstract

For a simple graph G, the energy $\mathcal{E}(G)$ is defined as the sum of the absolute values of all the eigenvalues of its adjacency matrix $A(G)$. Let n, m, respectively, be the number of vertices and edges of G. One well-known inequality is that $\mathcal{E}(G) \leq \lambda_1 + \sqrt{(n-1)(2m-\lambda_1)}$, where λ_1 is the spectral radius. If G is k-regular, we have $\mathcal{E}(G) \leq k + \sqrt{k(n-1)(n-k)}$. Denote $\mathcal{E}_0 = k + \sqrt{k(n-1)(n-k)}$. Balakrishnan [*Linear Algebra Appl.* **387** (2004) 287–295] proved that for each $\epsilon > 0$, there exist infinitely many n for each of which there exists a k-regular graph G of order n with $k < n-1$ and $\frac{\mathcal{E}(G)}{\mathcal{E}_0} < \epsilon$, and proposed an open problem that, given a positive integer $n \geq 3$, and $\epsilon > 0$, does there exist a k-regular graph G of order n such that $\frac{\mathcal{E}(G)}{\mathcal{E}_0} > 1 - \epsilon$. In this paper, we show that for each $\epsilon > 0$, there exist infinitely many such n that $\frac{\mathcal{E}(G)}{\mathcal{E}_0} > 1 - \epsilon$. Moreover, we construct another class of simpler graphs which also supports the first assertion that $\frac{\mathcal{E}(G)}{\mathcal{E}_0} < \epsilon$.

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1 Introduction

Let G be a simple graph with n vertices and m edges. Denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of G. Note that λ_1 is called the spectral radius. The energy of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. For more information on graph energy we refer to [5, 6], and for terminology and notations not defined here, we refer to Bondy and Murty [2].

One well-known inequality for the energy of a graph G is that $\mathcal{E}(G) \leq \lambda_1 + \sqrt{(n-1)(2m-\lambda_1)}$. If G is k-regular, we have $\mathcal{E}(G) \leq k + \sqrt{k(n-1)(n-k)}$. Denote $\mathcal{E}_0 = k + \sqrt{k(n-1)(n-k)}$. In [1], Balakrishnan investigated the energy of regular graphs and proved that for each $\epsilon > 0$, there exist infinitely many n for each of which there exists a k-regular graph G of order n with $k \leq n-1$ and $\frac{\mathcal{E}(G)}{\mathcal{E}_0} < \epsilon$. In this paper, we construct another class of simpler graphs which also support the above assertion. Furthermore, we show that for each $\epsilon > 0$, there exist infinitely many n satisfying that there exists a k-regular graph G of order n with $k < n-1$ and $\frac{\mathcal{E}(G)}{\mathcal{E}_0} > 1-\epsilon$, which answers the following open problem proposed by Balakrishnan in [1]:

Open problem. Given a positive integer $n \geq 3$ and $\epsilon > 0$, does there exist a k-regular graph G of order n such that $\frac{\mathcal{E}(G)}{\mathcal{E}_0} > 1-\epsilon$ for some $k < n-1$?

2 Main results

Throughout this paper, we denote $V(G)$ the vertex set of G and $E(G)$ the edge set of G. Firstly, we will introduce the following useful result given by So et al. [3].

Lemma 1 *Let $G-e$ be the subgraph obtained by deleting an edge e of $E(G)$. Then*

$$\mathcal{E}(G) \leq \mathcal{E}(G-e) + 2.$$

We then formulate the following theorem by employing the above lemma.

Theorem 1 ([1]) *For any $\varepsilon > 0$, there exist infinitely many n for each of which there exists a k-regular graph G of order n with $k < n-1$ and $\mathcal{E}(G)/\mathcal{E}_0 < \varepsilon$.*

Proof. Let $q > 2$ be a positive integer. We take q copies of the complete graph K_q. Denote by $v_1, \ldots, v_q$ the vertices of K_q and the corresponding vertices in each copy by $v_1[i], \ldots, v_q[i]$, for $1 \leq i \leq q$. Let G_{q^2} be a graph consisting of q copies of K_q and q^2 edges by joining vertices $v_j[i]$ and $v_j[i+1]$, $(1 \leq i < q)$, $v_j[q]$ and $v_j[1]$ where $1 \leq j \leq q$. Obviously, the graph G_{q^2} is $q+1$ regular. Employing Lemma 1, deleting all the q^2 edges joining two copies of K_q, we have $\mathcal{E}(G_{q^2}) \leq \mathcal{E}(qK_q) + 2q^2$. Thus, $\mathcal{E}(G_{q^2}) \leq 2q(q-1) + 2q^2$. Then, it follows that

$$\begin{aligned}\frac{\mathcal{E}(G_{q^2})}{\mathcal{E}_0} &\leq \frac{4q^2 - 2q}{q + 1 + \sqrt{(q+1)(q^2-1)(q^2-q-1)}} \\ &\leq \frac{4q^2 - 2q}{(q^2-q-1)\sqrt{q+1}} \to 0 \text{ as } q \to \infty.\end{aligned}$$

Thus, for any $\varepsilon > 0$, when q is large enough, the graph G_{q^2} satisfies the required condition. The proof is thus complete. ∎

Theorem 2 *For any $\varepsilon > 0$, there exist infinitely many n satisfying that there exists a k-regular graph of order n with $k < n - 1$ and $\mathcal{E}(G)/\mathcal{E}_0 > 1 - \varepsilon$.*

Proof. It suffices to verify an infinite sequence of graphs satisfying the condition. To this end, we focus on the Paley graph (for details see [4]). Let $p \geq 11$ be a prime and $p \equiv 1(mod\ 4)$. The Paley graph G_p of order p has the elements of the finite field $GF(q)$ as vertex set and two vertices are adjacent if and only if their difference is a nonzero square in $GF(q)$. It is well known that the Paley graph G_p is a $(p-1)/2$-regular graph. And the eigenvalues are $\frac{p-1}{2}$ (with multiplicity 1) and $\frac{-1\pm\sqrt{p}}{2}$ (both with multiplicity $\frac{p-1}{2}$). Consequently, we have

$$\mathcal{E}(G_p) = \frac{p-1}{2} + \frac{-1+\sqrt{p}}{2} \cdot \frac{p-1}{2} + \frac{1+\sqrt{p}}{2} \cdot \frac{p-1}{2} = (p-1)\frac{1+\sqrt{p}}{2} > \frac{p^{3/2}}{2}.$$

Moreover, $\mathcal{E}_0 = \frac{p-1}{2} + \sqrt{\frac{p-1}{2}(p-1)(p-\frac{p-1}{2})}$, we can deduce that

$$\mathcal{E}(G_p)/\mathcal{E}_0 > \frac{\frac{p^{3/2}}{2}}{\frac{p-1}{2}(\sqrt{p+1}+1)} > \frac{\frac{p^{3/2}}{2}}{\frac{p}{2}(\sqrt{p}+2)} \to 1 \text{ as } p \to \infty.$$

Therefore, for any $\varepsilon > 0$ and some integer N, if $p > N$, it follows that $\mathcal{E}(G_p)/\mathcal{E}_0 > 1 - \varepsilon$. The theorem is thus proved. ∎

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